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Structures of Opposition in Fuzzy Rough Sets

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Abstract. The square of opposition is as old as logic. There has been a recent renewal of interest on this topic, due to the emergence of new structures (hexagonal and cubic) extending the square. They apply to a large variety of representation frameworks, all based on the notions of sets and relations. After a reminder about the structures of opposition, and an introduction to their gradual extensions (exemplified on fuzzy sets), the paper more particularly studies fuzzy rough sets and rough fuzzy sets in the setting of gradual structures of opposition.

Keywords: square of opposition; fuzzy set; fuzzy relation; rough set.

1. Introduction

Fuzzy set theory [41, 42, 44, 45, 46] and rough set theory [29, 31, 34, 33, 32] are two important frameworks which have been introduced and developed in the second half of the previous century, and which have been proved to be very successful in information processing. They are both mathematically based on the notions of sets and relations, but are motivated by quite different concerns, although they can be (somewhat artificially) related [30], and the idea of granulation [43] can be encountered in both settings.

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While fuzzy set theory makes the notion of membership to a class gradual and softens equivalence relations into similarity relations, rough set theory bounds, from above and below, any subset of elements in terms of equivalence classes of indiscernible elements (having the same attribute values). Since their respective concerns are orthogonal rather than competing, it makes sense to consider different forms of hybridizations of the two theories, as pointed out quite early [17, 18]; see also [15].

Due to their mathematical nature based on sets and relations, the two theories have established connections with logic [26, 16, 14]. Thus it should not come as a surprise that they can be considered in the perspective of the square of opposition. The square of opposition is a representation of different forms of opposition arising among four logical statements. It has been introduced by Aristotle and then studied throughout the centuries, in particular by Middle-Age logicians. Then, it has been forgotten by modern logic, until its interest was rediscovered by Robert Blanché in relation with cognitive modeling concerns [8], in the second half of XXth century. In the last past years, it has raised again a lot of interest [3, 5, 6, 7] and it has been extended in several ways, generating new structures of opposition, which can be displayed on hexagons, or cubes, in particular. Two generic instantiations of the cube of opposition are in terms of intersections of sets and of compositions of relations respectively [21], which explains the universality of this structure in knowledge representation.

These structures can indeed be encountered in different fields including artificial intelligence-related areas [19, 1, 21]. In particular, oppositions in rough sets have been studied, which arise from approximations, relations, attributes [10, 40, 11]. Recently, a gradual extension of the square, of the hexagon and of the cube of oppositions has been proposed [20, 21]. So, it seems natural to apply these new structures to fuzzy rough sets and rough fuzzy sets [17, 18]. This is the purpose of this paper.

The paper is organized as follows. Section 2 provides an introduction to structures of opposition, and their gradual extensions, then exemplified by the case of fuzzy sets. Section 3 studies oppositions in fuzzy rough sets, while section 4 deals with rough fuzzy sets.

2. Structures of opposition: The Boolean case

In this section we introduce the basic structures of opposition, and then their gradual extensions. For an overview of the square of opposition and generalized geometric representation of opposition we refer to [3, 4, 19].

2.1. Square, Hexagon, and Cube of Opposition

The traditional square of opposition involves four related logical statements with different quantifiers and the classical negation operation $\neg$. Given a statement $p(x)$, the four corners read as $\mathbf{A}$: $\forall x\ p(x)$, $\mathbf{E}$: $\forall x\ \neg p(x)$, $\mathbf{I}$: $\exists x\ p(x)$, $\mathbf{O}$: $\exists x\ \neg p(x)$. Let us notice that we suppose the existence of some x such that $p(x)$ holds, for avoiding existential import problems. A graphical representation of the square is usually given in Figure 1.

Clearly, these four corners are not independent from each other. The links among them can be highlighted by interpreting **A**, **I**, **E**, and **O** as the truth values of the statements, that is as Boolean variables. So, we have (see, e.g., [27]):

(a) **A** and **O** are the negation of each other, as well as E and I. In a logical reading $\mathbf{A} \equiv \neg\mathbf{O}$ and $\mathbf{E} \equiv \neg\mathbf{I}$.

(b) **A** entails **I**, and **E** entails **O**, i.e., vertical arrows represent *implication* relations $\mathbf{A} \to \mathbf{I}$ and $\mathbf{E} \to \mathbf{O}$.

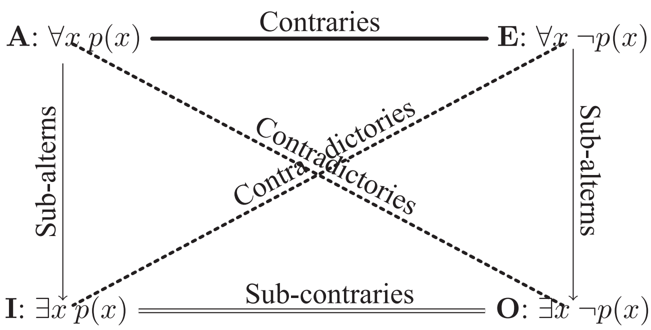

Figure 1. Square of opposition

(c) **A** and **E** cannot be true together, but may be false together: $\neg$**A** $\vee$ $\neg$**E** should hold (they are in a *contrariety* relation).

(d) **I** and **O** cannot be false together, but may be true together: **I** $\vee$ **O** should hold (they are in a *subcontrariety* relation).

Moreover, the above conditions are not independent. Several links can be established among them, which have to be considered when generalizing the square to the gradual case:

(Dep 1) Conditions **(a)(b) imply condition (c).** That is, $\neg$**A** $\vee$ $\neg$**E** is a consequence of **A** $\equiv$ $\neg$**O** and **E** $\to$ **O** (or of **E** $\equiv$ $\neg$**I** and **A** $\to$ **I**) in the square.

(Dep 2) **Conditions (a)(b) imply condition (d).** That is, **I** $\vee$ **O** is a consequence of **A** $\equiv$ $\neg$**O** and **A** $\to$ **I** (or of **E** $\equiv$ $\neg$**I** and **E** $\to$ **O**).

(Dep 3) **Conditions (a)(c) imply conditions (b)(d)** and **conditions (a)(d) imply conditions (b)(c): A** $\equiv$ $\neg$**O**, **E** $\equiv$ $\neg$**I**, together with $\neg$**A** $\vee$ $\neg$**E** entail **A** $\to$ **I**, **E** $\to$ **O** and **I** $\vee$ **O**. Similarly, **A** $\equiv$ $\neg$**O**, **E** $\equiv$ $\neg$**I**, together with **I** $\vee$ **O** entail **A** $\to$ **I**, **E** $\to$ **O** and $\neg$**A** $\vee$ $\neg$**E**.

The hexagon of opposition [8, 4] is built on the square by considering the union of **A**, **I** obtaining **U**, and the conjunction of **E**, **O** obtaining **Y** (see Figure 2). It was then noticed that the six corners define three squares of opposition: the one we start with **AIEO**, but also **AYOU** and **EYIU**.

Besides, the square of opposition can be generalized to a cube of opposition. The cube of opposition has the four corners **A**, **I**, **E**, and **O** in the front facet and four other corners in the back, namely, **a**, **i**, **e**, and **o**. In Figure 3, a Boolean cube is represented with the statements corresponding to the new back corners. This cube was first introduced by Reichenbach [37] in a systematic discussion of syllogisms, and rediscovered in [19]. It is worth mentioning that the vertices of the diagonal squares **AaoO** and **EeiI** are related by a Klein group of transformations applied to logical statements, first identified by Piaget [35]. For instance, $R(\mathbf{A}) = C(N(\mathbf{A})) = \mathbf{a}$, $C(\mathbf{O}) = N(R(\mathbf{O})) = \mathbf{a}$, or $N(R(C(\mathbf{E}))) = N(R(\mathbf{i})) = N(\mathbf{I}) = \mathbf{E}$, where i) $I(\phi) = \phi$ (identity), $N(\phi) = \neg\phi$ (negation), $R(\phi) = f(\neg p, \neg q, ...)$ (reciprocation), and $C(\phi) = \neg f(\neg p, \neg q, ...)$ (correlation). It can be easily checked that $N = RC$, $R = NC$, $C = NR$, and $I = NRC$. See Figure 3 and [19].

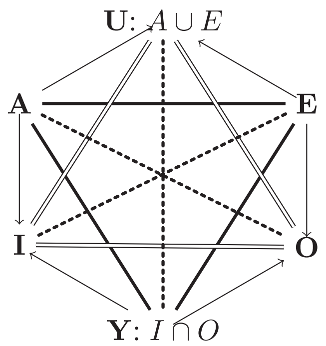

Figure 2. Hexagon of opposition

2.2. The Cube of Rough Sets

Several kinds of opposition structures can be defined in the rough set context: based on relations, approximations, or attributes [10, 40, 11]. Here we are interested in the cube based on upper and lower approximations. It is well known that a rough set is a pair of lower $L_R(A)$ and upper $U_R(A)$ approximations of a subset A of a set X defined according to a relation R and such that $L_R(A) \subseteq U_R(A)$. More precisely, given an approximation space (X, R) with R a binary relation on R, the two approximations are defined as [38]:

$$L_R(A) = \{x \in X | xR \subseteq A\}$$
$$U_R(A) = \{x \in X | xR \cap A \neq \emptyset\}$$

where xR is the neighborhood of x with respect to R, that is $xR = \{y|xRy\}$. These two sets are at the basis of a square of oppositions: $L_R(A)$ is the corner **A** and $U_R(A)$ the corner **I**. The other two corners are obtained by complementation: $L_R^c(A)$ is corner **O** and $U_R^c(A)$ corner **E** (this last set is also known as the exterior of A). The usual interpretation attached to these sets is that the lower approximation contains the objects surely belonging to A, the exterior contains objects surely not belonging to A and the remaining objects form the boundary. So, once we extend the square into a hexagon, the top corner contains the totality of objects on which we are certain, namely, $L_R(A) \cup U_R^c(A)$ whereas the bottom one contains the objects on which we are totally undecided: $U_R(A) \setminus L_R(A)$.

Now, when moving to the cube, we distinguish two cases, depending on whether L and U are dual to each other, that is $L(A) = U^c(A^c)$, or not. In this last case, a cube can be defined using as back square the approximations applied to A^c: $L_R(A^c)$, $U_R(A^c)$, $L_R^c(A^c)$ and $U_R^c(A^c)$ are respectively the corners **a, i, o, e**. On the other hand, if the lower and upper approximations are dual, the front and back squares collapse.

However, another kind of cube can be defined by considering a so called sufficiency operator:

$$[[A]]_R := \{x \in X | A \subseteq xR\} = \{x \in X | y \in A \Rightarrow xRy = \bigcap_{x \in A} xR\}$$

and the dual operator: $\ll A \gg_R = \{x \in X | A \cup xR \neq \emptyset\}$. The whole cube arising from lower, upper and sufficiency approximation is drawn in Figure 4. Note that $[[A]]_R$ can be equivalently written as $L_{R^c}(A^c)$.

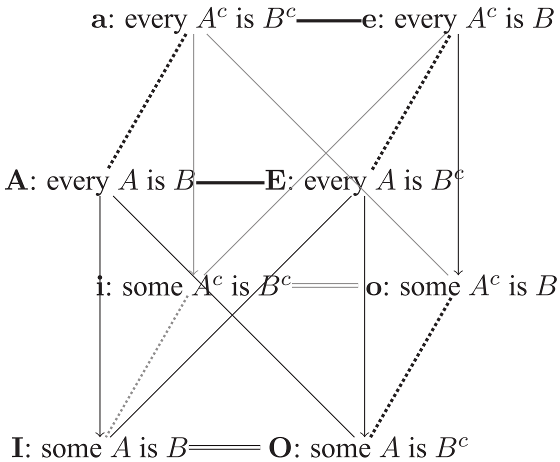

Figure 3. Cube of opposition

In other words, the back facet of this cube is the same as the front facet where the relation R is replaced by its complement. However R has a lot of properties, usually, while its complement does not have them. So even if $L_{R^c}(A^c)$ is formally the lower approximation of A^c with respect to R^c, it often hardly stands as a genuine lower approximation.

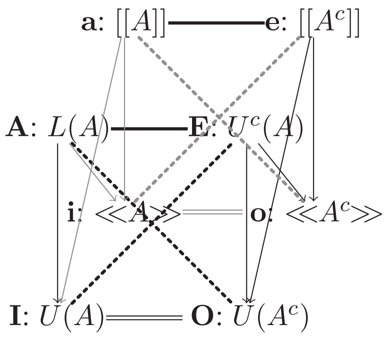

Figure 4. Cube of oppositions induced by rough approximations

If R is an equivalence relation with equivalence classes $C_i, i = 1, \dots p$, then $[[A]]_R := C_i$ if $A \subseteq C_i$, and $\emptyset$ otherwise, which indicates that this notion is not very fruitful in that case. If R is only symmetric and reflexive, then R can still be written as $\bigcup_{i=1,\dots,p} C_i \times C_i$, where C_i is maximal such that $C_i \times C_i \subseteq R$, but the C_i's may overlap. This amounts to saying that an undirected graph is the union of its maximal cliques. Then, $xR = \bigcup_{i:x\in C_i} C_i$ and $[[A]]_R = \bigcap_{x\in A} \bigcup_{i:x\in C_i} C_i$. Interestingly we may have that $[[A]]_R \cap A = \emptyset$. For instance, assume that $p = 2$, C_1 and C_2 overlap, and $A = (C_1 \setminus C_2) \cup (C_2 \setminus C_1)$, then $[[A]]_R = C_1 \cap C_2$. Note that $[[A]]_R$ contains all those elements related to all elements in A. So, $[[A]]_R$ can be viewed as all bridges that make all elements in A communicate.

Remark 2.1. In modal logic, the standard necessity operator expresses the fact that a property is a necessary condition for some other properties to hold. Moreover, Kripke semantics is given through possible worlds and a binary relation R connecting them. In this standard environment, the idea of a sufficient condition has no place and, further, Kripke semantics cannot account for irreflexive relations. The sufficiency operator is introduced in order to overcome these deficiencies [25, 23]. These ideas were then borrowed by data analysis and the above operator $[[\cdot]]$ introduced as an approximation operator [28, 22].

3. Gradual structures of opposition

In this section, we try to extend the Boolean structure of opposition such as the one in Figure 3 to the case where sets are fuzzy, so that statements appearing on the vertices are true to a degree between 0 and 1. As we shall see, one difficulty we shall meet is due to the fact that the strong link between entailment (relating vertices **A** and **I**, or yet **E** and **O**, for instance) and negation may be lost. In particular, we have that $p \to q \equiv \neg p \vee q$, and negation is $\neg p = p \to \perp$ (where $\perp$ denotes the contradiction) and is involutive. However, in the gradual setting the negation as defined above is generally not involutive. As the square of opposition heavily relies on the involutivity property [12], the design of gradual squares, hexagons and cubes of opposition becomes more tricky.

3.1. The Gradual Square

The gradual square of opposition associates a degree in $[0, 1]$ to each corner. Let us name the degree of corners **A**, **I**, **E**, and **O** respectively as $\alpha, \iota, \epsilon, o$. Then, we outline two possibilities for generalizing the square: the weak and the strong one. In order to do it, we need an involutive negation n, a commutative conjunction $*$, the dual disjunction $\oplus$ and an implication denoted by $s \Rightarrow t$.

The connectives we are going to consider are based on standard operations on $[0, 1]$:

- *Negations* n are unary functions such that $n(0) = 1$ and $n(1) = 0$. A negation is said *involutive* if $\forall x, n(n(x)) = x$.
- *Commutative conjunctions*, i.e., binary operations $* : [0, 1]^2 \mapsto [0, 1]$ such that $x * y = y * x$; $0 * x = 0$; $1 * x = x$. In particular, *triangular norms* (*t–norms*) $*$, are associative and monotonic commutative conjunctions. Given a conjunction $*$ and an involutive negation n, the dual disjunction is defined by De Morgan properties as $x \oplus y = n(n(x) * n(y))$. The dual of a t-norm is named *triangular conorm* (t–conorm).
- *Implications* $\to$, i.e. a binary function on $[0, 1]$ such that $1 \to 0 = 0$ and $1 \to 1 = 0 \to 1 = 0 \to 0 = 1$. It is said to be a *border implication* if $\forall x \in [0, 1]$, $1 \to x = x$. Particular border implications are the *residual* of a left-continuous t-norm, defined as $x \to_* y := \sup\{x \in [0, 1] : x * z \leq y\}$. Another important class is the one of *strong implications* (S-implications): given a conjunction $*$ and an involutive negation n, a strong implication is defined as $x \Rightarrow_S y := n(x * n(y)) = n(x) \oplus y$ where $\oplus$ is the dual of $*$.

The *strong form of the gradual square of opposition* requires that the above constraints (a)–(b) are encoded as follows:

(a) **A** and **O** are the negation of each other, as well as **E** and **I**: $\alpha = n(o)$ and $\epsilon = n(\iota)$

(b) The implication is assumed to be a strong one, i.e., $s \Rightarrow t = n(s * n(t)) = n(s) \oplus t$. Then, **A** entails **I**, and **E** entails **O** is modeled as $\alpha \Rightarrow \iota = 1$ and $\epsilon \Rightarrow o = 1$, i.e., $\alpha * n(\iota) = 0$ and $\epsilon * n(o) = 0$;

(c) **A** and **E** cannot be true together, but may be false together. It can be encoded by $\alpha * \epsilon = 0$ or equivalently $n(\alpha * \epsilon) = 1$;

(d) **I** and **O** cannot be false together, but may be true together. It can be encoded by $n(\iota) * n(o) = 0$ or equivalently $n(n(\iota) * n(o)) = 1$, i.e. $\iota \oplus o = 1$.

The *weak form* of the gradual square differs on condition (b), requiring only that $\alpha \leq \iota$ and $\epsilon \leq o$.

In case of the *strong form*, dependencies (Dep1)–(Dep3) still hold given the four conditions (a)–(d). On the other hand, this is not the case for the weak form, so further contraints have to be considered if we desire to have a complete faithful extension of the square to the gradual case. For instance, we can require the conjunction $*$ to be a nilpotent t-norm and n to be the standard involutive negation $n(x) = 1 - x$.

3.2. Gradual Cube

In case of the gradual cube, degrees $\alpha', \iota', \epsilon', o'$ are also associated to corners **a**, **i**, **e**, and **o** of the cube, with the requirement to form a weak/strong gradual square of opposition. That is, the conditions on the front and back squares (strong form) are:

(a) $\alpha = n(o)$, $\epsilon = n(\iota)$ and $\alpha' = n(o')$, $\epsilon' = n(\iota')$;

(b) $\alpha * n(\iota) = 0$, $\epsilon * n(o) = 0$ and $\alpha' * n(\iota') = 0$, $\epsilon' * n(o') = 0$;

(c) $\alpha * \epsilon = 0$ and $\alpha' * \epsilon' = 0$;

(d) $n(\iota) * n(o) = 0$ and $n(\iota') * n(o') = 0$.

Moreover, we have some constraints on the side facets (these conditions derive from analogous ones holding in the Boolean cube, see [12]):

(e) $\alpha * n(\iota') = 0$, that is ***A*** entails ***i***;

(f) $\alpha' * n(\iota) = 0$, ***a*** entails ***I***;

(g) $\epsilon' * n(o) = 0$, ***e*** entails ***O***;

(h) $\epsilon * n(o') = 0$, ***E*** entails ***o***.

which are equivalent to the conditions that we have to require on the top and bottom facets:

(i) $\alpha' * \epsilon = 0$, which means that ***a*** and ***E*** cannot be true together;

(j) $\alpha * \epsilon' = 0$, ***A*** and ***e*** cannot be true together;

(k) $n(\iota') * n(o) = 0$, that is ***i*** and ***O*** cannot be false together;

(l) $n(\iota) * n(o') = 0$, ***I*** and ***o*** cannot be false together.

In case of the weak form of the square, while conditions (a), (c) and (d) are left unchanged, the conditions (b) become $\alpha \leq \iota$, $\epsilon \leq o$ and $\alpha' \leq \iota'$, $\epsilon' \leq o'$, whereas, the side (top/bottom) facets conditions read as:

(e') $\alpha \leq \iota'$;

(f') $\alpha' \leq \iota$;

(g') $\epsilon' \leq o$;

(h') $\epsilon \leq o'$.

3.3. Gradual Hexagon

Finally, the gradual hexagon of opposition is built from the square by considering the union of **A**, **I** obtaining **U** with degree ν and the conjunction of **E**, **O** obtaining **Y** with degree γ). That is, we define $\nu = \alpha \oplus \epsilon$ and $\gamma = \iota * o$.

Since the six corners define three squares of opposition: the standard one **AIEO**, then **AYOU** and **EYIU**, we have to impose the conditions (a)–(d) on them. In the case of the hexagon, we are going to consider only the weak form of the square since the strong form would require that $*$ is a nilpotent t-norm, hence satisfying all the weak form constraints plus the dependency ones (Dip1)–(Dip3). So, the four constraints on the squares **AYOU** and **EYIU** imply the following:

(a) $\nu = n(\epsilon)$. This is true by definition of ν. Indeed, $\nu = \alpha \oplus \epsilon = n(\iota) \oplus n(o) = n(\iota * o) = n(\gamma)$.

(b) A entails U and Y entails O, that is $\alpha \leq \nu$ and $\gamma \leq o$. Again by definition, this means $\alpha \leq \alpha \perp \epsilon$ and $\iota * a \leq o$, which is true for any choice of monotonic conjunction $*$ and disjunction $\oplus$, and in particular for all triangular norms and triangular co-norms.
Similarly, we have to require that Y entails I, E entails U, i.e., $\gamma \leq \iota$ and $\epsilon \leq \nu$.

(c) $\alpha * \gamma = 0$ and $\epsilon * \gamma = 0$. This condition, generally, does not follow from the previous ones, so we should impose it.

(d) $n(o) * n(\nu) = 0$ and $n(i) * n(\nu) = 0$. This condition is equivalent to the previous one.

As discussed in [12], sufficient conditions for all these constraints to hold are that condition (c) hold and $*$,$\oplus$ are dual norm and co-norm or that $*$ is a nilpotent triangular norm, such as $\alpha * \beta = \max(0, \alpha + \beta - 1)$.

3.4. Example: The Cube of Fuzzy Sets

Going back to the cube of Figure 3, the entailments of the top facet may be rewritten in terms of empty intersections of sets of objects A, B, and their complements A^c, B^c, while the bottom facets refer to non empty intersections, as pointed out in [21]. See Figure 5. Note that we assume $A \neq \emptyset$, $A^c \neq \emptyset$, $B \neq \emptyset$, and $B^c \neq \emptyset$ here, for avoiding the counterpart of the existential import problems, since now the sets A and B play symmetric roles in the statements associated to the vertices of the cube.

This cube extends to the case where A and B are normalized fuzzy subsets of X, e.g., $A : X \mapsto [0, 1]$. We denote degrees of membership by $A(x), B(x), \ldots$ Suppose we use the $\min$-based and $1 - (\cdot)$-based definitions of intersection and complementation respectively. Then $\iota = \sup_x \min(A(x), B(x))$, and

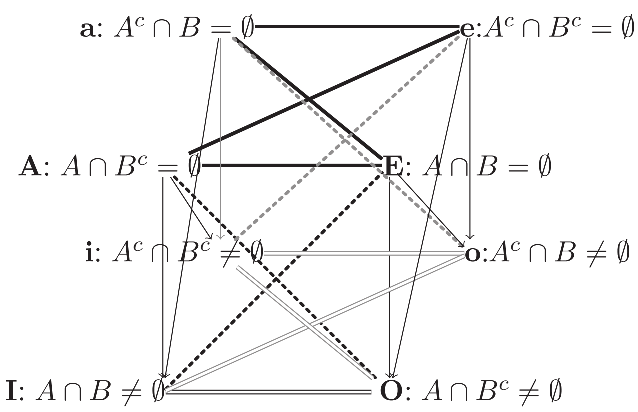

Figure 5. Cube of opposition of set intersection indicators

$o = \sup_x \min(A(x), 1 - B(x))$; $\alpha = 1 - o$ and $\epsilon = 1 - \iota$. Then, it can be checked that $n(\iota) * n(o) = 0$, or equivalently $\iota \oplus o = 1$, namely,

$$\sup_x \min(A(x), B(x)) + \sup_x \min(A(x), 1 - B(x)) \geq B(x_0) + 1 - B(x_0) = 1,$$

where $A(x_0) = 1$ (normalization of A). From which it follows by duality that $\alpha * \epsilon = 0$, and we have $\alpha = \inf_x \max(1 - A(x), B(x)) \leq \iota = \sup_x \min(A(x), B(x))$ if A is normalized. The other conditions of the cube can be checked as well (provided that A^c, B, B^c are also normalized).

4. Opposition in fuzzy rough sets

In this section we first recall basic notions of fuzzy rough sets, i.e., approximations of fuzzy sets induced by a fuzzy relation.

4.1. Fuzzy Rough Sets

As basic definition of fuzzy rough set, we consider the one given in [36] generalized to any kind of fuzzy relation. At first, we need some definitions on fuzzy sets. Let X be the universe of investigation. A fuzzy binary relation is a mapping $R : X \times X \mapsto [0, 1]$ and R is said

serial	iff	$\forall x \in X, \exists y \in X : R(x, y) = 1$
reflexive	iff	$\forall x \in X : R(x, x) = 1$
symmetric	iff	$\forall x, y \in X : R(x, y) = R(y, x)$

Then, we can define fuzzy rough sets.

Definition 4.1. [36] Given a t-norm $*$, a fuzzy binary relation R on a universe X, an implication $\rightarrow$, then the lower and upper approximations of a fuzzy set A are:

$$L_R(A)(x) := \inf_{y \in X} \{R(x, y) \rightarrow A(y)\} \tag{1}$$

$$U_R(A)(x) := \sup_{y \in X} \{R(x, y) * A(y)\} \tag{2}$$

A fuzzy rough set is the pair $(L_R(A), U_R(A))$.

To qualify as a genuine rough set, this pair must obey some requirements

- $L_R(A) \subseteq U_R(A)$, a sufficient condition being that $R(x,y) \to A(y) \le R(x,y) * A(y)$, for some y. If we assume R is serial, then take y s.t. $R(x,y)=1$, which yields $1 \to A(y) \le A(y)$, which holds if we use a border implication [9]. The stronger natural condition $L_R(A) \subseteq A \subseteq U_R(A)$ also requires a reflexive fuzzy relation.

- the duality condition $U_R(A) = n(L_R(n(A)))$ holds if there is an involutive negation n such that $n(\sup_{y\in X} R(x,y) * n(A(y))) = \inf_{y\in X} R(x,y) \to A(y)$, which means that the implication verifies $a \to b = n(a * n(b))$ so that $n(a) = a \to 0$. These conditions restrict the choice of the pair $(*, \to)$ (for instance Łukasiewicz conjunction and implication connectives, or yet minimum and Kleene-Dienes implication).

In the next subsection, we study the gradual square, cube and hexagon that the approximations in fuzzy rough sets originate.

4.2. Square from Approximations

Given a fuzzy set A, its lower and upper approximations with their complement (with respect to an involutive negation n) can generate the standard square of opposition **A**: $L_R(A)$, **I**: $U_R(A)$, **E**: $n(U_R(A))$, **O**: $n(L_R(A))$, where $n(A)$ is the membership function of the complement of fuzzy set A.

Of course, conditions (a)–(d) have to be satisfied and they read as :

(a) $\alpha = n(o) \equiv L_R(A)(x) = n(n(L_R(A)(x))$ and $\epsilon = n(\iota) \equiv n(U_R(A)(x)) = n(U_R(A)(x))$.

(b) $\alpha * n(\iota) = L_R(A)(x) * n(U_R(A)(x) = 0$ in case of the strong form of the square and $\alpha \le \iota \equiv L_R(A)(x) \le U_R(A)(x)$, $\epsilon \le o \equiv n(U_R(A)(x)) \le n(L_R(A)(x))$ in case of the weak form.

(c) $\alpha * \epsilon = 0 \equiv L_R(A)(x) * n(U_R(A)(x)) = 0$.

(d) $n(\iota) * n(o) = 0$

Proposition 4.2.

1. Condition (a) is always true whenever n is involutive.
2. In case of the strong form, condition $\alpha * n(\iota) = L_R(A)(x) * n(U_R(A)(x)) = 0$ is sufficient to derive the other conditions.
3. In case of the weak form, a sufficient condition for (b) is to have R serial and $\to$ a border implication and n order reversing.
4. Condition (d) is an immediate consequence of (a) and (c).

Proof:

1. It follows by definition.

2. By construction we have $\alpha = n(o)$ from which we can derive the other conditions (all dependencies Dep1–Dep3 hold in case of the strong form of the square).

3. From seriality of R and the fact that $\rightarrow$ is a border implication we get $L_R(A)(x) \leq U_R(A)(x)$ [9]. Then, if n is order reversing, we easily get $n(U_R(A)(x)) \leq n(L_R(A)(x))$.

□

We notice that the seriality of R is a standard condition in order to obtain a square of opposition by a relation [11, 12].

Condition (c) is usually neglected in fuzzy rough set approaches. However, it seems quite natural and important to require that the lower approximation and the exterior region ($n(U_R(A))$) are disjoint. Moreover, from point (2) of the above proposition, it plays an important role and it imposes some constraints on the definition of the fuzzy set A and the fuzzy relation R. For instance, it straightforwardly holds that:

Proposition 4.3. A sufficient condition for $\alpha * n(\iota)$ to be zero is that either $L_R(A)(x)$ or $n(U_R(A))(x)$ are equal to zero. That is:

$$\forall x \exists y : \quad R(x,y) \rightarrow A(y) = 0 \quad \text{or} \quad R(x,y) * A(y) = 1$$

However, these conditions are seldom applicable since they rarely occur. For instance, the second one, due to the properties of the t-norm, comes down to requiring that $R(x,y) = A(y) = 1$, while $R(x,y)$ and $A(y)$ are independent quantities. In the general case, it is not so obvious to impose some constraints on the t-norm $*$ and the implication $\rightarrow$ to make $\alpha * n(\iota) = 0$ hold for all possible values x. Further investigations both in theory and on case studies are needed in this direction.

A similar square of opposition can be obtained with other kinds of fuzzy rough approximations, for instance, the *loose* and *tight* ones defined as follows [13].

Definition 4.4. Let R be a fuzzy binary relation on X and f a fuzzy set on X. The *tight* approximation of f is defined as

$$\forall y \in X \quad L_t(A)(y) = \inf_{z \in X}\{R_z(y) \rightarrow \inf_{x \in X}\{R_z(x) \rightarrow A(x)\}\}$$
$$\forall y \in X \quad U_t(A)(y) = \sup_{z \in X}\{R_z(y) * \sup_{x \in X}\{R_z(x) * A(x)\}\}$$

The *loose* approximation of f is defined as

$$\forall y \in X \quad L_l(A)(y) = \sup_{z \in X}\{R_z(y) * \inf_{x \in X}\{R_z(x) \rightarrow A(x)\}\}$$
$$\forall y \in X \quad U_l(A)(y) = \inf_{z \in X}\{R_z(y) \rightarrow \sup_{x \in X}\{R_z(x) * A(x)\}\}$$

Assuming that R is a similarity relation, i.e., it is reflexive and symmetric, we can prove the following relationship with the standard lower and upper approximations:

$$\text{(loose)} \quad L_l(A) = U_R(L_R(A)) \quad U_l(A) = U_R(U_R(A))$$
$$\text{(tight)} \quad L_t(A) = L_R(L_R(A)) \quad U_t(A) = L_R(U_R(A))$$

Hence, due to the monotonicity of L_R it easily follows that, provided R is reflexive and symmetric, $L_l(A) \subseteq U_l(A)$ and $L_t(A) \subseteq U_t(A)$. So, both the tight and loose approximations, together with their complement with respect to an order reversing negation, can build a weak form of gradual square of opposition. Of course, the further constraints $L_l(A) * n(U_l(A)) = 0$ and $L_t(A) * n(U_t(A)) = 0$ have to be satisfied.

4.3. The Gradual Cube of Approximations

Extending the square of fuzzy rough approximations to a cube leads to several possibilities to explore. As in the Boolean setting, the front and back of the cube coincide in case of dual approximations. If the lower and upper approximations are not dual (that is $L(A) \neq n(U(n(A)))$), we can define a cube considering the approximations applied to the complement of A. This possibility has been also discussed in [10] with respect to Boolean rough sets (see Section 2.2).

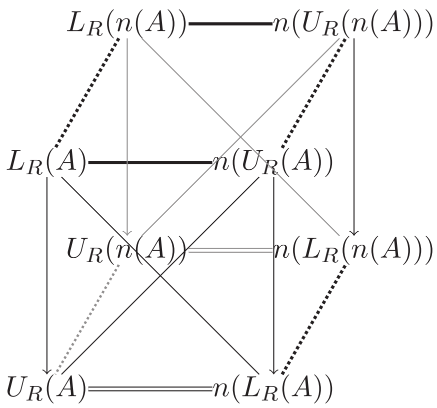

Figure 6. Cube of opposition from non-dual fuzzy rough sets

The consideration on conditions (a)–(d) on the back square are the same as before, since it is the same square of the front applied to a different set. If we wish to respect also the side and top conditions, once a order reversing negation is considered, they reduce (weak form) to the following two: $L_R(A) \subseteq U_R(n(A))$ and $L_R(n(A)) \subseteq U_R(A)$. It is not an easy task to give general conditions under which these two conditions hold together. Indeed, we can give examples that do not satisfy them even if in presence of crisp relations with strong properties.

Example 4.5. Let R be an equivalence relation on two objects x, y such that it always assumes the value 1 and define the set $A(x) = A(y) = 0.6$. Then, $L_R(A)(x) = L_R(A)(y) = 0.6 \geq U(n(A))(x) = U(n(A))(y) = 0.4$. On the other hand, considering the same relation and the set $B(x) = B(y) = 0.4$, we get $L_R(n(B))(x) = L_R(n(B))(y) = 0.6 \geq U(A)(x) = U(A)(y) = 0.4$. However note that in this example A and B are not normal, which may create existential import problems.

Finally, if we consider the Klein group of the four Piaget transformations already mentioned, namely: identity $I(\phi) = \phi$; negation $N(\phi) = \neg\phi$; reciprocation $R(\phi) = f(\neg p, \neg q, \ldots)$ and correlation $C(\phi) = \neg f(\neg p, \neg q, \ldots)$, and if we consider the two squares (visualized in Figure 6) obtained from the diagonals of the cube, i.e., those with vertices $(L_R(A), L_R(n(A)), n(L_R(A)), n(L_R(n(A)))$ and with vertices

$(U_R(A), U_R(n(A)), n(U_R(A)),\ n(U_R(n(A)))$, we see that these vertices are still exchanged by this Klein group as in the Boolean case, provided that n is involutive.

4.4. Cube from Approximations and Sufficiency Operator

Whether upper and lower approximations are dual or not, another kind of cube can be defined as an extension of the cube of relations defined in [11]. In this case, the back square is built starting from a *sufficiency* operator and its dual. So, we have to introduce a new kind of "approximation" in fuzzy rough sets (as well as its dual) based on a fuzzy sufficiency operator.

Definition 4.6. Let R be a fuzzy relation and A a fuzzy set, the fuzzy set of *bridge points* of A and its dual, are respectively defined as:

$$[[A]]_R(x) := \inf_y \{A(y) \rightarrow R(x, y)\} \tag{3}$$

$$\ll A \gg_R(x) := n[[n(A)]] \tag{4}$$

The set $[[A]]_R$ corresponds to the corner (***a***) of the cube, whereas $\ll A\gg_R$ to (***i***). The value $[[A]](x)$ can be interpreted as the degree to which x is related to the set A. If $[[A]](x) = 1$ then $A \subseteq xR$. More precisely, $[[A]]_R(x)$ may be understood as the extent to which x is connecting all elements in A, since it estimates if any y in A is (highly) related to x in the sense of R. In other words, to what extent any element in A can communicate through x. In $[[A]]_R(x)$, the implication is reversed with respect to $L_R(A)(x)$. In case we take the conjunction of both, namely, $L_R(A)(x) \wedge [[A]]_R(x)$, we get an estimate that may represent how much x is R-similar to A, namely $A \sim_R x$. Note that $\sim_R$ is not transitive, but serial. On the other hand, $\ll A \gg_R(x)$ is the degree of non-relationship of x with elements in $n(A)$. The fact that $\ll A \gg_R(x) = 0$ can be interpreted as x is in relation with all the elements in $n(A)$. However, another option for defining $\ll A\gg_R(x)$ in the spirit of Equation (2), such as $\ll A \gg = \sup_y n(A(u)) * n(R(x, y))$, might be worth investigating.

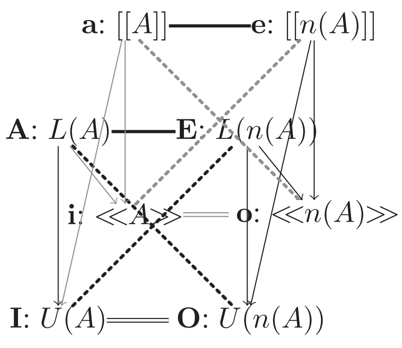

Figure 7. Cube of opposition induced by fuzzy-rough sets and sufficiency operator

Conditions (a)–(d) on the back square read as:

(a) $[[A]]_R = n(\ll n(A)\gg_R)$ and $[[n(A)]]_R = (\ll A\gg_R)^c$ hold by definition;

(b) The two conditions $\alpha' \leq \iota'$ and $\epsilon' \leq o'$ are equivalent and require that the sufficiency operator implies its dual: $[[A]]_R \subseteq \ll A \gg_R$. A standard requirement in the analogous Boolean case is to ask for seriality of the fuzzy relation $n(R)$. In this case, it means to require that for all x there exists an element y such that $R(x, y) = 0$. So the condition is satisfied, if $n(\neg(n(A))) \subseteq n(A)$ where $\neg$ is the negation obtained by the implication used to define the sufficiency operator $[[\cdot]]$. For instance, this holds for a residual implication induced by a t-norm without non-trivial zero divisors [24] and any involutive negation n, indeed in this case it holds $\neg x \leq n(x)$.

(c) $[[A]]_R * [[n(A)]]_R = 0$. Similarly to the front square, it is not easy to give general conditions for this constraint to hold.

(d) due to duality of $\ll n(A) \gg_R$ and $[[A]]_R$, it is the same as condition (c).

Now, let us consider conditions on side/bottom faces. They read as

- $L_R(A) \subseteq \ll n(A) \gg_R$. With a similar reasoning as in point (b) above, a sufficient condition for this constraint to hold is to have at least one element such that $A(y) = 0$ and $\rightarrow$ to be a residual implication induced by a t-norm without non-trivial zero divisors.

- $[[A]] \subseteq U_R(A)$. In this case, it is sufficient to have a border implication and a normalized fuzzy set, i.e., there should exist a value y such that $A(y) = 1$.

4.5. Hexagon

As discussed in section 3 a hexagon is defined considering the conjunction of ***A***, ***E*** and the disjunction of I, O. In this case, they read as

$$
\begin{array}{ll}
(\boldsymbol{U}) & L_R(\boldsymbol{A}) \oplus n(\boldsymbol{U}_R(A)) \\
(\boldsymbol{Y}) & \boldsymbol{U}_R(\boldsymbol{A}) * n(L_R(\boldsymbol{A}))
\end{array}
$$

We stress that it is not necessary that $*$ and $\oplus$ are the operators used to define the approximations L and U.

By analogy with the Boolean case [10], (***U***) represents what we know with certainty on A, indeed $L_R(A)$ are the elements surely belonging to A (to a certain degree in this fuzzy case) whereas $n(U_R(A))$ are those surely not belonging to A. On the other hand, (***Y***) represents the total uncertainty region, representing the elements belonging to the possibility region $U_R(A)$ but not to the certainty one $L_R(A)$, that is, to the boundary.

Following section 3, in order to get a hexagon, besides the conditions on the square, one of the two conditions is sufficient

- $*$ is a nilpotent t-norm;

- $*, \oplus$ are dual t-norm and t-conorm and conditions $\alpha * \gamma = \epsilon * \gamma = 0$ hold. In this case, they read as $L_R(A) * U_R(A) * n(L_R(A)) = 0$ and $n(U_R(A)) * U_R(A) * n(L_R(A)) = 0$.

4.6. Special Cases

Up to now, we have considered an extension of classical rough sets using a fuzzy relation and a fuzzy set (see Definition 4.1). We now investigate what happens when either the relation or the set are fuzzy and the other is crisp.

4.6.1. Crisp Set and Fuzzy Relation

Let A be a subset of the universe X and R a fuzzy relation on X. In order to approximate A given the knowledge expressed by R, a first and immediate solution is to apply the same definitions of fuzzy rough sets to a crisp set. So, equations in Definition 4.1 become now:

$$L_R(A)(x) := \begin{cases} 1 & A = X \\ \inf_{y \in A^c}\{\neg R(x,y)\} & A \neq X \end{cases} \tag{5}$$

$$U_R(A)(x) := \begin{cases} 0 & A = \emptyset \\ \sup_{y \in A}\{R(x,y)\} & A \neq \emptyset \end{cases} \tag{6}$$

where $\neg$ is the negation operator induced by the implication. Being a particular case of fuzzy rough sets, all the considerations formulated in the previous section, apply also here. Moreover, some constraints are here always (or more often) satisfied. At first let us notice that the following result holds by definition of L_R and U_R:

Lemma 4.7. If R is

- *Serial* then for all x either $L_R(A)(x) = 0$ or $U_R(A)(x) = 1$;
- *Reflexive* then for all $x \notin A$ we have $L_R(A)(x) = 0$ and for all $x \in A$, $U_R(A)(x) = 1$.

So, considering that R should be serial, we have that

Proposition 4.8. Condition (c) of the square holds: for all x, $L_R(A)(x) * n(U_R(A)(x)) = 0$.

In case of the cube of opposition and non-dual approximations[1], for the condition on the side and bottom faces we can state that

Proposition 4.9. If R is reflexive then $\inf_{y \in A^c}\{\neg R(x,y)\} \leq \sup_{y \in A^c}\{R(x,y)\}$.

That is, reflexivity of R is a sufficient condition to make the condition on side and bottom face holds.

Now, in this context, the sufficiency operator becomes:

$$[[A]]_R(x) := \begin{cases} 1 & A = \emptyset \\ \inf_{y \in A}\{R(x,y)\} & A \neq \emptyset \end{cases}$$

So, also for the sufficiency operator and its dual, we have that

[1]Let us notice that it can happen more frequently than in the general case that the approximations are dual, that is: $L_R(A) = nU_R(nA)$. For instance if $\neg x = n(x) = 1 - x$.

Proposition 4.10. The seriality condition on $n(R)$ implies the side condition $[[A]]_R \subseteq \ll A \gg_R$. Moreover, if A is not empty also the other side condition $[[A]]_R \subseteq U_R(A)$ holds.

Proof:
By definition of the sufficiency operator and due to the fact that A is Boolean, either $[[A]]_R(x) = 0$ or $\ll A \gg_R (x) = 1$, and then trivially $[[A]]_R \subseteq \ll A \gg_R$. Also, $[[A]]_R \subseteq U_R(A)$ follows easily by definition. $\square$

We remark that, in general, we suppose that A is not empty since, in the classical square, the existence of some x such that $p(x)$ holds is assumed (see Section 3).

Finally, in case of the hexagon, we see that

Proposition 4.11. If the relation R is reflexive, then conditions $\alpha * \gamma = 0$ and $\epsilon * \gamma = 0$ hold.

So, in order to have a full realization of the hexagon, it is sufficient to consider a reflexive relation and a pair of dual t-norm and t-conorm.

Another possible way to define approximations in case of a crisp set and a fuzzy relation is to consider α-cuts of the fuzzy relation [39]. That is, given R we consider the family of relations

$$R_\alpha(x,y) = \begin{cases} 0 & R(x,y) < \alpha \\ 1 & R(x,y) \geq \alpha \end{cases}$$

In this case, we obtain a family of classical approximation spaces and if R is a fuzzy equivalence (max-min transitive) relation, R_α are all (Boolean) equivalence relations [39]. So, for each R_α, we can compute the standard approximations and obtain a family of classical square/cube/hexagon of opposition [10].

4.7. Fuzzy Set and Crisp Equivalence Relation

In case of R crisp and A fuzzy, *rough fuzzy sets* [17, 18] can be defined as

$$L_R(A)(x) = \inf\{A(y) | y \in [x]_R\}$$
$$U_R(A)(x) = \sup\{A(y) | y \in [x]_R\}$$

where $[x]_R$ is the equivalence class of x relatively to the relation R. It can be easily seen that these two equations are special cases of equations in Definition 4.1 whenever R can assume only values $0, 1$ and we use a border implication. So, the general conditions of Section 4 apply also here and for some constraints can be simplified as follows.

In case of the square, we have that $L_R(A)(x) \leq U_R(A)(x)$ is always true. Then, when extending the square to the cube we have that L and U are dual operators: given an involutive negation n, then $L_R(A) = n(U(n(A)))$. So, we can only consider the cube built from the sufficiency operator, which reads as: $[[A]]_R(x) = \inf\{\neg A(y) | y \notin [x]_R\} = L_{R^c}(\neg A)(x)$. Conditions on the back and side square do not simplify further in this case with respect to what described in section 4.4. The same can be said in the case of the hexagon.

As in the previous case, another approach is to use α-cuts, in this case to build a family of sets approximating the given fuzzy set [39]. Let A be a membership function of a fuzzy set, then for any α-cut A_α of A we can define its classical approximations $(L(A_\alpha), U(A_\alpha))$ and so obtain a family of structures of opposition, based on (Boolean) rough sets [10].

5. Conclusion

Opposition structures are a powerful tool to express all properties of rough sets and fuzzy rough sets with respect to negation in a synthetic way. After having studied the structure of opposition in Boolean rough sets [10] and extended the notion of square, cube and hexagon of opposition to the graded case [20, 12], we studied here the geometric representation of oppositions in the setting of fuzzy rough sets, that is when the basic elements of the approximations, the relation and the sets, are fuzzy. As particular cases also the situation where either the relation or the sets are crisp have been investigated. In all these situations we describe how to obtain at first a square of opposition and then extended structures, such as the cube and the hexagon. This study has stressed the importance of the relation between the inner and exterior regions of a set: they should be disjoint, a constraint always neglected. As an open problem, we leave it to the future a deeper study on the conditions on the fuzzy relation R and on the operations $*, \rightarrow$ to obtain the satisfaction of this constraint. We also introduced the sufficiency operator (and its dual) in fuzzy rough sets. The usefulness of this new operator in applications is yet to be explored. Finally, results in this study extend beyond the field of fuzzy rough sets and could be useful in fuzzy formal concept analysis [2].

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